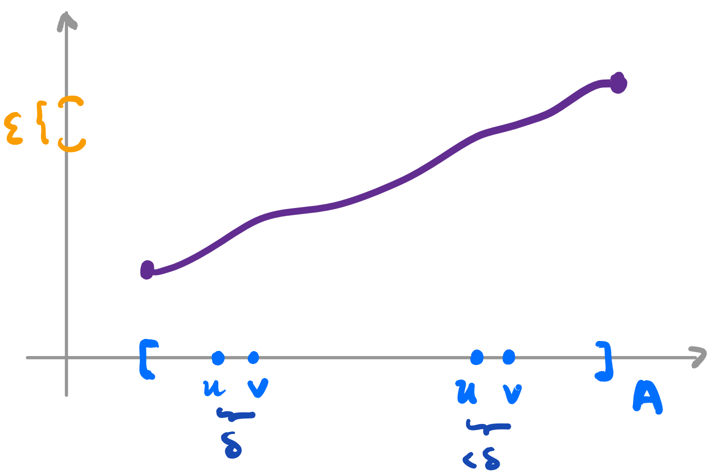
MATH 2050C Lecture 23 (Apr 15)

[Last Problem Set 12 posted, due on Apr 23.]

Last time: Uniform continuity

Def^n: Let $f: A \to \mathbb{R}$ be a function. We say f is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(u) - f(v)| < \varepsilon \quad \text{whenever } u, v \in A,\ |u - v| < \delta$$



Example: $f(x) = x$, $x \in [0,1]$

Non-Example: $f(x) = \frac{1}{x}$, $x \in (0,1]$

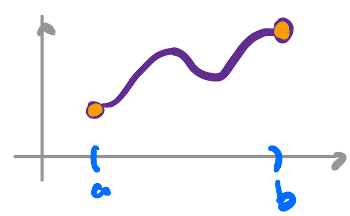
$f(x) = \sin\frac{1}{x}$, $x \in (0,1]$

Last time: non-uniform continuity criteria

Uniform Continuity Thm: $f: [a,b] \to \mathbb{R}$ cts $\Rightarrow$ uniformly cts on $[a,b]$

Continuous Extension Thm:

If $f: (a,b) \to \mathbb{R}$ is uniformly cts on (a,b), then $\exists!$ cts extension $\bar{f}: [a,b] \to \mathbb{R}$



WANT: $\bar{f}(a) = \lim_{x \to a} f(x)$

$\bar{f}(b) = \lim_{x \to b} f(x)$

Lemma: Let $f : A \to \mathbb{R}$ be uniform cts.

(x_n) in A Cauchy seq. $\Rightarrow$ $(f(x_n))$ in $\mathbb{R}$ Cauchy seq.

Proof of Continuous Extension Thm:

It suffices to show the existence of $\lim_{x \to a} f(x)$, $\lim_{x \to b} f(x)$, then we can define $\bar{f} : [a,b] \to \mathbb{R}$ as

$$\bar{f}(x) := \begin{cases} f(x), & x \in (a,b) \\ \lim_{x \to a} f(x), & x = a \\ \lim_{x \to b} f(x), & x = b \end{cases}$$

Claim: $\lim_{x \to a} f(x)$ exists.

Pf: By Sequential Criteria, it suffices to prove that

$\exists\, L \in \mathbb{R}$ s.t. for $\underline{\underline{\text{ANY}}}$ seq. (x_n) in (a,b) s.t.

$\lim (x_n) = a$ we have $\lim (f(x_n)) = L$

Step 1: Find one such L.

Choose $x_n := a + \frac{1}{n}$ $\forall n \in \mathbb{N}$ (defined when n is large)

Note: $(x_n) \to a$ hence is Cauchy

By Lemma, $(f(x_n))$ is Cauchy, hence converging to some $L \in \mathbb{R}$.

Step 2: Show that the L we obtained in Step 1 works for **ALL** seq. $(x_n') \to a$ ((x_n') in (a,b)).

Take an arbitrary seq (x_n') in (a,b) converging to a.

[Idea: $\underset{\text{close}}{x_n \approx x_n'} \ \overset{\text{unif.}}{\underset{\text{cts}}{\Longrightarrow}} \ \underset{\text{close}}{f(x_n) \approx f(x_n')}$]

Since $\lim(x_n) = a = \lim(x_n')$, we have

$$\lim |x_n - x_n'| = 0 \qquad \text{by Limit theorem}$$

To see $(f(x_n')) \to L$. Suppose, by Step 1, $(f(x_n')) \to L'$

Let $\varepsilon > 0$. By uniformly continuity of f, $\exists\ \delta = \delta(\varepsilon) > 0$

(*)...... s.t. $|f(u) - f(v)| < \varepsilon$ when $u, v \in (a,b)$, $|u - v| < \delta$

Now, $\lim |x_n - x_n'| = 0 \Rightarrow \exists\ K = K(\delta) \in \mathbb{N}$ s.t.

$$|x_n - x_n'| < \delta \qquad \forall\, n \geq K$$

Hence, we have from (*),

$$|f(x_n) - f(x_n')| < \varepsilon \qquad \forall\, n \geq K$$

Take $n \to \infty$, we obtain $|L - L'| \leq \varepsilon$ but $\varepsilon > 0$ is arbitrary. Then, we have $L = L'$.

$\square$

Picture: